Study material for first semester NEP(Major)
Department of Mathematics

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Calculus & Analytical Geometry

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Contents

1 Hyperbolic functions and its application

We are already familiar with trigonometric function. There are another function looking like the trigonometric function which is known as the hyperbolic functions. The hyperbolic function is a combination of exponential function and there are similar formula like the trigonometry functions. In this unit we define the three main hyperbolic functions and sketch their graphs. We also discuss some identities regarding the hyperbolic function, inverse and reciprocal functions.

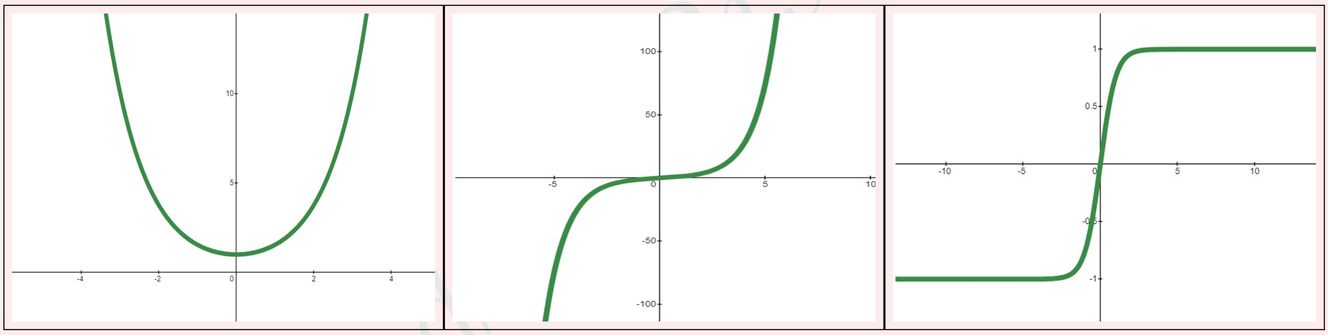
Definition 1.1. *The hyperbolic functions coshx and sinhx are defined by*

$$coshx = \frac{e^x + e^{-x}}{2} \qquad\qquad sinhx = \frac{e^x - e^{-x}}{2}$$

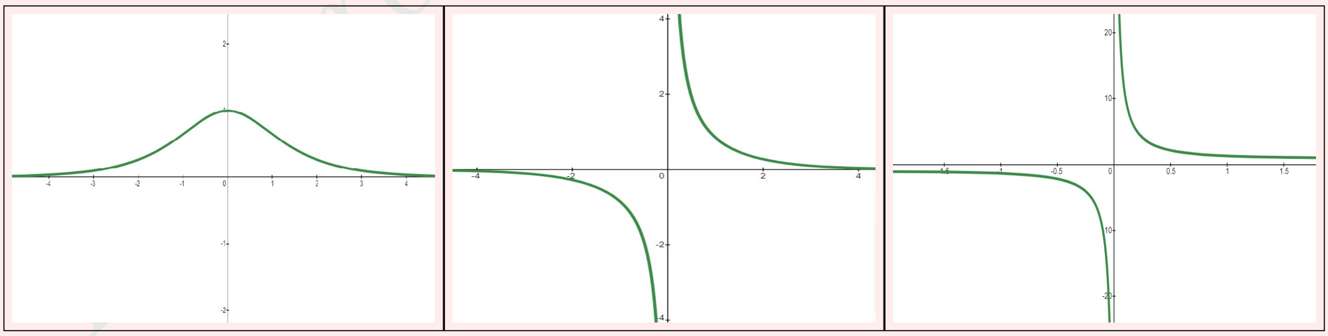
Using the above definition we write

$$tanhx = \frac{sinhx}{coshx} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad cosechx = \frac{1}{sinhx} = \frac{2}{e^x - e^{-x}}$$
$$cothx = \frac{coshx}{sinhx} = \frac{e^x + e^{-x}}{e^x - e^{-x}}, \quad sechx = \frac{1}{coshx} = \frac{2}{e^x + e^{-x}}$$

The graph of the above functions are shown below:



(a) $y = \cosh x$ (b) $y = sinhx$ (c) $y = tanhx$



(d) $y = sechx$ (e) $y = cosechx$ (f) $y = cothx$

Figure 1: The different type of Hyperbolic functions.

1.1 Formula related to hyperbolic function

There are some formula for the hyperbolic function some of them are shown below

Theorem 1.2. *The following identities are important.*

1. $cosh^2x - sinh^2x = 1$

2. $sinh(2x) = 2sinh(x)cosh(x)$

3. $cosh(2x) = cosh^2x + sinh^2x$

4. $sinh(x+y) = sinhxcoshy + sinhycoshx$

5. $cosh(x+y) = coshxcoshy + sinhxsinhy$

6. $cosh^2\left(\frac{x}{2}\right) = \frac{1+coshx}{2}$

7. $sinh^2\left(\frac{x}{2}\right) = \frac{coshx-1}{2}$

Proof. 1. It is clear from the definition that

$$
\begin{aligned}
cosh^2x - sinh^2x =& \left(\frac{e^x+e^{-x}}{2}\right)^2 - \left(\frac{e^x-e^{-x}}{2}\right)^2 \\
=& e^x.e^{-x} \\
=& 1
\end{aligned}
$$

2. Again we see

$$
\begin{aligned}
sinh(2x) &= \frac{e^{2x}-e^{-2x}}{2} \\
&= \frac{(e^x+e^{-x})(e^x-e^{-x})}{2} \\
&= 2\frac{(e^x-e^{-x})}{2}\frac{(e^x+e^{-x})}{2} \\
&= 2sinh(x)cosh(x)
\end{aligned}
$$

3. Using the definition of $coshx$ we write,

$$
\begin{aligned}
cosh(2x) &= \frac{e^{2x}+e^{-2x}}{2} \\
&= \frac{2(e^{2x}+e^{-2x})}{4} \\
&= \frac{2((e^x)^2+(e^{-x})^2)}{4} \\
&= \frac{(e^x+e^{-x})^2+(e^x-e^{-x})^2}{4} \\
&= \left(\frac{e^x+e^{-x}}{2}\right)^2 + \left(\frac{e^x-e^{-x}}{2}\right)^2 \\
&= cosh^2(x) + sinh^2(x)
\end{aligned}
$$

4. From the definition of hyperbolic function it is clear that

$$\begin{aligned}
sinh(x+y) &= \frac{e^{x+y}-e^{-(x+y)}}{2}\\
&= \frac{(e^x e^y - e^{-x}e^{-y})}{2}\\
&= \frac{2(e^x e^y - e^{-x}e^{-y})}{4}\\
&= \frac{(e^x-e^{-x})(e^y+e^{-y})+(e^x+e^{-x})(e^y-e^{-y})}{4}\\
&= \frac{(e^x-e^{-x})}{2}\frac{(e^y+e^{-y})}{2}+\frac{(e^x+e^{-x})}{2}\frac{(e^y-e^{-y})}{2}\\
&= sinhxcoshy + coshxsinhy
\end{aligned}$$

5. Similar to the previous.

6. Adding first and third identity and get $2cosh^2x = 1 + cosh(2x)$.
   Now replace x by $\frac{x}{2}$ and taking simple calculation we obtain $cosh^2\left(\frac{x}{2}\right) = \frac{1+cosh(x)}{2}$.

7. This proof is similar to the above and so we ignore it.

□

1.2 Derivative of Hyperbolic function

The derivative of hyperbolic functions are as follows:

(i) $\frac{d}{dx}(coshx) = sinhx$ (ii) $\frac{d}{dx}(sinhx) = coshx$ (iii) $\frac{d}{dx}(tanhx) = sech^2x$

(iv) $\frac{d}{dx}(sechx) = -sechxtanhx$ (v) $\frac{d}{dx}(cosechx) = -cosechxcothx$ (vi) $\frac{d}{dx}(cothx) = -cosech^2x$

Justification:

(i) From the definition of derivative of exponential function and hyperbolic function we see

$$\frac{d}{dx}(coshx) = \frac{d}{dx}\left(\frac{e^x+e^{-x}}{2}\right) = \left(\frac{e^x-e^{-x}}{2}\right) = sinhx$$

(ii) Similarly,

$$\frac{d}{dx}(sinhx) = \frac{d}{dx}\left(\frac{e^x-e^{-x}}{2}\right) = \left(\frac{e^x+e^{-x}}{2}\right) = coshx$$

(iii) Therefore using above derivative we write,

$$\frac{d}{dx}(tanhx) = \frac{d}{dx}\left(\frac{sinhx}{coshx}\right)$$
$$= \left(\frac{coshx\frac{d}{dx}(sinhx) - sinhx\frac{d}{dx}(coshx)}{cosh^2x}\right)$$
$$= \left(\frac{coshxcoshx - sinhxsinhx}{cosh^2x}\right)$$
$$= \left(\frac{cosh^2x - sinh^2x}{cosh^2x}\right) = \frac{1}{cosh^2x} = sech^2x$$

(iv)

$$\frac{d}{dx}(sechx) = \frac{d}{dx}\left(\frac{1}{coshx}\right)$$
$$= \left(\frac{-1}{cosh^2x}\right)\frac{d}{dx}(coshx)$$
$$= \left(\frac{-sinhx}{cosh^2x}\right)$$
$$= -\,sechxtanhx$$

(v) This is similar to (iv).

(vi) This is similar to (iii).

1.3 Inverse of Hyperbolic functions

From the graphs of the hyperbolic functions it is clear that all of them have inverses within some range. The corresponding domain and range are shown below:

Function	Domain	Range
$cosh^{-1}x$	$[1, \infty)$	$[0, \infty)$
$sinh^{-1}x$	$(-\infty, \infty)$	$(-\infty, \infty)$
$tanh^{-1}x$	$(-1, 1)$	$(-\infty, \infty)$
$sech^{-1}x$	$(0, 1]$	$[0, \infty)$
$cosech^{-1}x$	$(-\infty, 0) \cap (0, \infty)$	$(-\infty, 0) \cap (0, \infty)$
$coth^{-1}x$	$(-\infty, -1) \cap (1, \infty)$	$(-\infty, 0) \cap (0, \infty)$

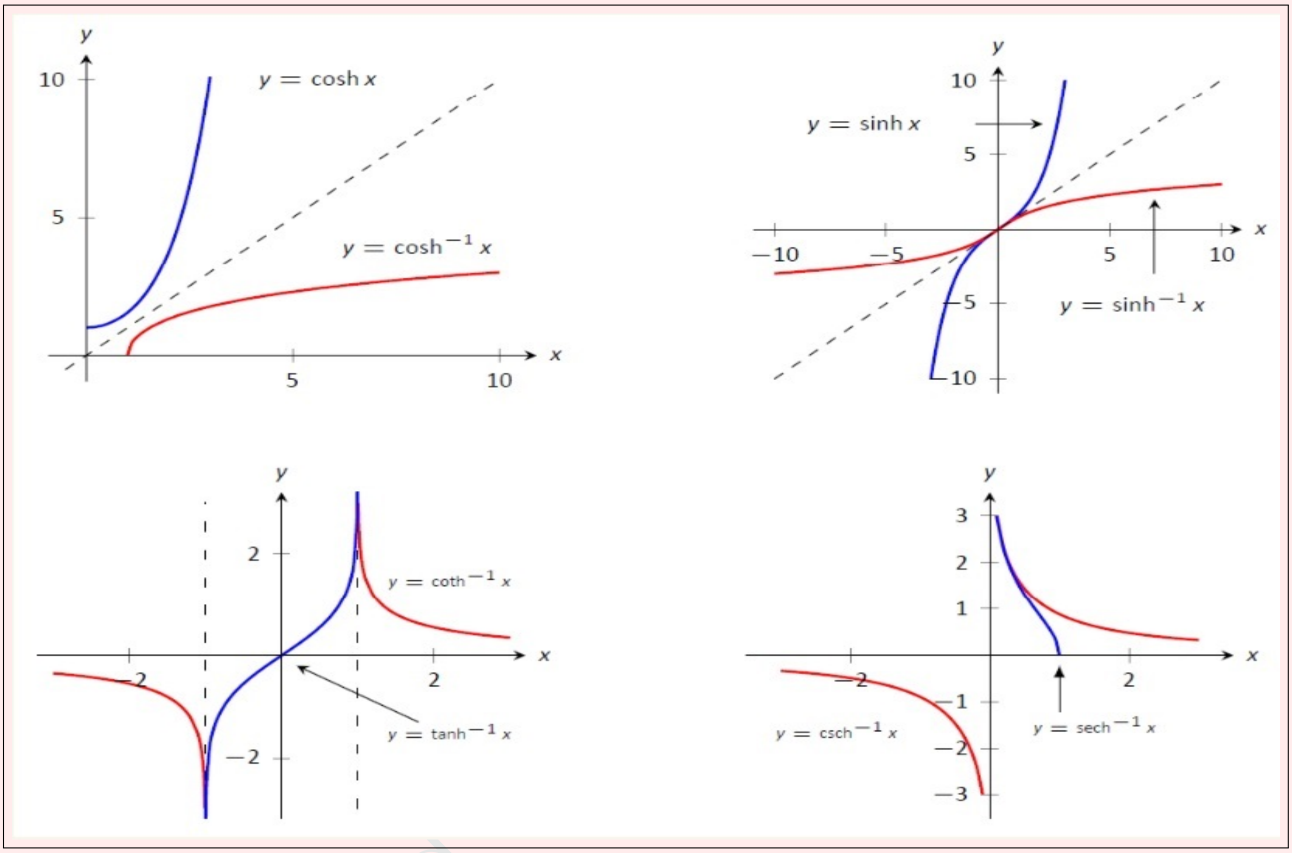
Definition 1.3. *The logarithmic definition of the inverse Hyperbolic functions are as follows:*

1. $cosh^{-1}x = ln(x + \sqrt{x^2 - 1}); x \geq 1$
2. $sinh^{-1}x = ln(x + \sqrt{x^2 + 1})$
3. $tanh^{-1}x = \frac{1}{2}ln\left(\frac{1+x}{1-x}\right); |x| < 1$
4. $sech^{-1}x = ln\left(\frac{1+\sqrt{1-x^2}}{x}\right); 0 < x \leq 1$

5. $cosech^{-1}x = ln\left(\frac{1}{x} + \frac{\sqrt{x^2+1}}{|x|}\right); x \neq 0$

6. $coth^{-1}x = \frac{1}{2}ln\left(\frac{x+1}{x-1}\right); |x| > 1$

The graph of the hyperbolic functions with the inverse as shown below:



(a)

Figure 2: The graph of some hyperbolic function and its inverse.

1.4 Derivative of Inverse Hyperbolic function

The derivative of inverse hyperbolic functions are as follows:

(i) $\frac{d}{dx}(cosh^{-1}x) = \frac{1}{\sqrt{x^2-1}}; x > 1$

(ii) $\frac{d}{dx}(sinh^{-1}x) = \frac{1}{\sqrt{x^2+1}}$

(iii) $\frac{d}{dx}(tanh^{-1}x) = \frac{1}{1-x^2}; |x| < 1$

(iv) $\frac{d}{dx}(sech^{-1}x) = -\frac{1}{x\sqrt{(1-x^2)}}; 0 < x < 1$

(v) $\frac{d}{dx}(cosech^{-1}x) = -\frac{1}{|x|\sqrt{(1+x^2)}}; x \neq 0$

(vi) $\frac{d}{dx}(coth^{-1}x) = -\frac{1}{x^2-1}; |x| > 1$

Justification:

(i) Let

$$\begin{aligned}
& y = cosh^{-1}x \\
\implies & x = coshy \\
& \text{Differentiating both sides w.r.t } y \text{ and get} \\
& \frac{dx}{dy} = \frac{d}{dy}(coshy) = sinhy \\
\implies & \frac{dy}{dx} = \frac{1}{sinhy} = \frac{1}{\sqrt{cosh^2y - 1}} = \frac{1}{\sqrt{x^2 - 1}}
\end{aligned}$$

(ii) This is similar to (i).

(iii) Let

$$\begin{aligned}
& y = tanh^{-1}x \\
\implies & x = tanhy \\
& \text{Differentiating both sides w.r.t } y \text{ and get} \\
& \frac{dx}{dy} = \frac{d}{dy}(tanhy) = sech^2y \\
\implies & \frac{dy}{dx} = \frac{1}{sech^2y} = \frac{1}{1 - tanh^2y} = \frac{1}{1 - x^2}
\end{aligned}$$

The remaining proof are similar to the above and so we omit it.

2 Pedal equation

Let C be a plane curve and O is a fixed point. Therefore the pedal equation of the curve is define as follows:

Definition 2.1. *The pedal equation of a curve is a relation between r and p where r is the distance from O to a point on C and p is the perpendicular distance from O to the tangent line to C at the point. The point O is called the pedal point and the values r and p are sometimes called the pedal coordinates of a point relative to the curve and the pedal point.*

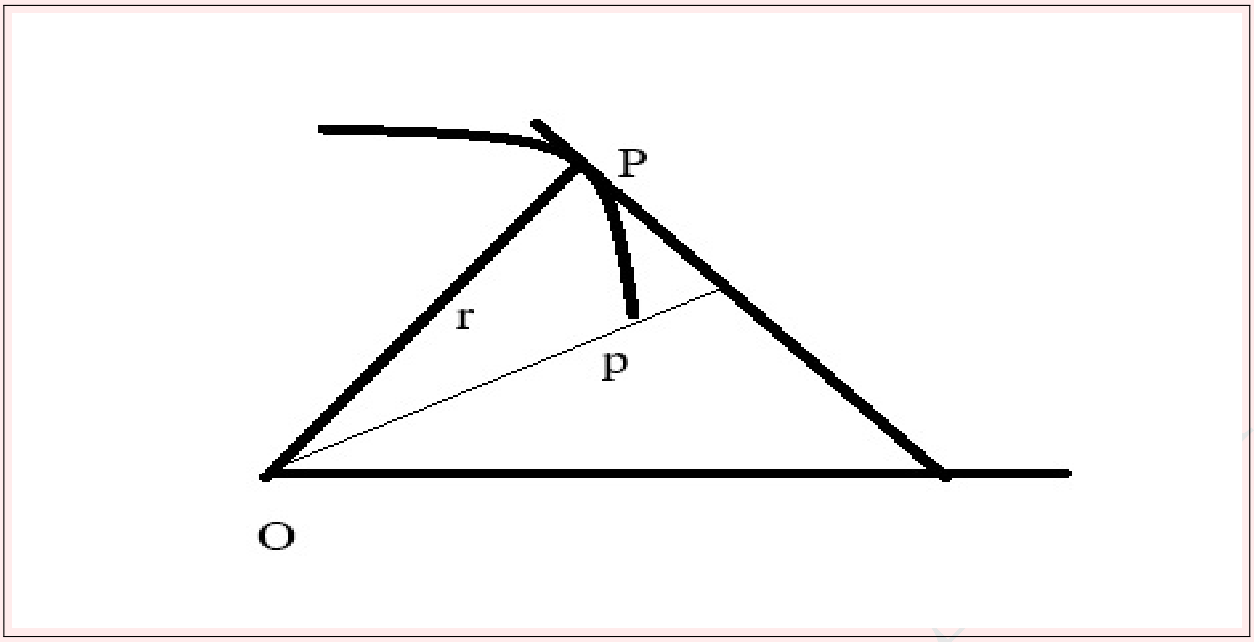
2.1 Pedal equation derived for Cartesian equation

Let the equation of the given curve be $f(x,y) = 0$. Also let the origin be the given point. Therefore the equation of the tangent at any point (x,y) is $Yf_y + Xf_x - (xf_x + yf_y) = 0$.
Then $p^2 = \dfrac{(xf_x + yf_y)}{(f_x^2 + f_y^2)}$ and $r^2 = x^2 + y^2$.
The pedal equation is the x, y elimination of above equations.

2.2 Pedal equation derived for Polar equation

If the given curve is of the form $r = f(\theta)$ and the given point be the pole then we know that $tan\phi = r\dfrac{d\theta}{dr}$ and $p = rsin\phi$.
The pedal equation for the polar curve is the θ eliminant of the above equations.

Note: The formula $\dfrac{1}{p^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$. may be used to find the pedal equation.



(a)

Figure 3: pedal of a curve C.

Example 1. *Find the pedal equation of the astroid* $x^{2/3} + y^{2/3} = a^{2/3}$.

Solution: The equation of tangent at any point (x, y) is $Xx^{-1/3} + Yy^{-1/3} = a^{2/3}$.
If p be the perpendicular distance from the origin then $p^2 = \dfrac{a^{4/3}}{x^{-2/3} + y^{-2/3}} = (axy)^{2/3}$.
Also it is known that $r^2 = x^2 + y^2 = (x^{2/3})^3 + (y^{2/3})^3 = (x^{2/3} + y^{2/3})^3 - 3x^{2/3}y^{2/3}(x^{2/3} + y^{2/3}) = a^2 - 3(axy)^{2/3}$
$\implies r^2 = a^2 - 3p^2$
$\implies r^2 + 3p^2 = a^2$
This is the required pedal equation.

3 Curvature

The term "Curvature" is used to discuss the measurement of bending of a curve in space. The definition of curvature of a curve at any point $P(x, y)$ is $\dfrac{d\psi}{ds}$, where s is the arc length and ψ is the angle between the initial line and the tangent of the curve at P. In general it is denoted by κ.

3.1 Radius of curvature

The radius of curvature (ρ) of a curve is defined as the reciprocal of the curvature i.e, $\rho = \dfrac{ds}{d\psi}$.
Formula for radius of convergence:

(a) For Cartesian equation $\rho = \dfrac{(1 + y_1^2)^{3/2}}{y_2}, (y_2 \neq 0)$, where suffix denotes the derivative w.r.t x.
Note: If the given function is of the form $x = f(y)$, then the above formula is replaced as $\rho = \dfrac{(1 + x_1^2)^{3/2}}{x_2}, (x_2 \neq 0)$, where suffix denotes the derivative w.r.t y.

(b) If the given function is of the form $f(x,y)=0$, then $\rho = \dfrac{(f_x^2+f_y^2)^{3/2}}{f_{xx}f_y^2 - 2f_{yx}f_xf_y + f_{yy}f_x^2}$.

(c) If the given function is of polar form i.e, $r=f(\theta)$, then $\rho = \dfrac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$.

(d) If the given function is of the intrinsic form i.e, $s=f(\psi)$, then $\rho = \dfrac{ds}{d\psi}$.

(e) If the curve is of the following form $x=f(t), y=g(t)$ then $\rho = \dfrac{f'(t)^2+g'(t)^{2^{3/2}}}{f'(t)g''(t)-g'(t)f''(t)}$.

(f) For pedal equation $p=f(r), \rho = r\dfrac{dr}{dp}$.

3.2 Center of curvature

The center of curvature of a point P of the curve is the limiting position of the point of intersection of the normal to the curve at P with the normal to the curve at a neighboring point Q on the curve as Q tends to P along the curve. The formula for coordinates of center of curvature is

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$
$$\bar{y} = y + \frac{(1+y_1^2)}{y_2}$$

3.3 Circle of curvature

The circle of curvature is the circle whose center is $(\bar{x},\bar{y})$ and the radius is ρ.

4 Asymptote

Definition 4.1. *An asymptote is a straight line or a curve that approaches a given curve as it heads toward infinity but never meets the curve. Such a pair of curves is called an asymptotic curve.*

The following schemetic diagram gives the idea about the asymptote of a curve. There exists two types of Asymptotes namely rectangular Asymptotes and oblique Asymptotes:

(i) Rectangular Asymptote: If an asymptote is parallel to x-axis or to y-axis, then it is called rectangular asymptote. There are two types of asymptotes:

  (a) Horizontal Asymptote: The Asymptote is a horizontal asymptote when x tends to ∞ or -∞, and the curve approaches some constant value b.

  (b) Vertical Asymptote: The asymptote is a vertical asymptote when x approaches some constant value c from left to right, and the curve tends to ∞ or -∞.

(ii) Oblique Asymptote: The asymptote is an oblique or slant asymptote when x moves towards infinity or –infinity and the curve moves towards a line $y=mx+b$. Here, m is not zero as in horizontal asymptote.

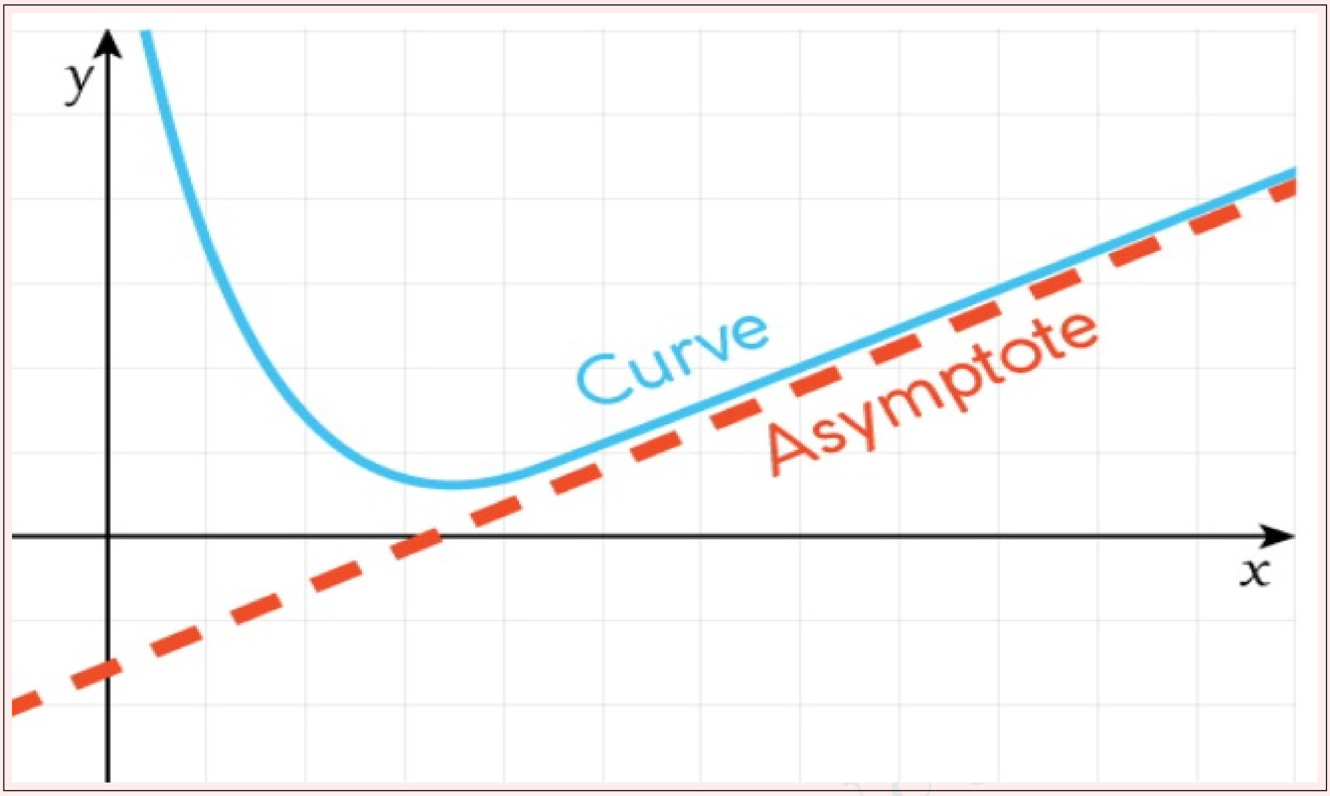


Figure 4: Asymptotes of a curve.

4.1 Method of finding rectangular asymptote

- To find an asymptote parallel to x-axis equate to zero the coefficient of highest power of x in the equation of the curve.
- To find an asymptote parallel to y-axis equate to zero the coefficient of highest power of y in the equation of the curve.

Example 2. *1. Find the asymptotes parallel to coordinate axes of the curve* $4x^2 + 9y^2 = x^2y^2$.
Solution: *The equation of the given curve is* $4x^2 + 9y^2 = x^2y^2$.
The coefficient of highest power of x^2 *is* $4 - y^2$.
Therefore the asymptotes parallel to x*-axis is given by the equation* $4 - y^2 = 0 \implies y = \pm 2$.
Hence $y = 2$ *and* $y = -2$ *are the two asymptotes parallel to* x*-axis Similarly, the coefficient of highest power of* y^2 *is* $9 - x^2$.
Therefore the asymptotes parallel to x*-axis is given by the equation* $9 - x^2 = 0 \implies x = \pm 3$.
Hence $x = 3$ *and* $x = -3$ *are the two asymptotes parallel to* y*-axis.*

4.2 Method of finding oblique asymptote:

Let us consider the following algebraic curve:

$$(a_0y^n + a_1y^{n-1}x + a_2y^{n-2}x^2 + ... + a_nx^n) + (b_0y^n + b_1y^{n-1}x + b_2y^{n-2}x^2 + ... + b_nx^n) + ... + (l_{n-1}y + l_nx) + k_n = 0.$$

The above equation can be written as $\phi_n(x, y) + \phi_{n-1}(x, y) + ... + \phi_1(x, y) + k_n = 0$, where $\phi_n(x, y)$ are homogeneous function of degree n. To find the oblique asymptotes of the form $y = mx + c$ consider the following steps:

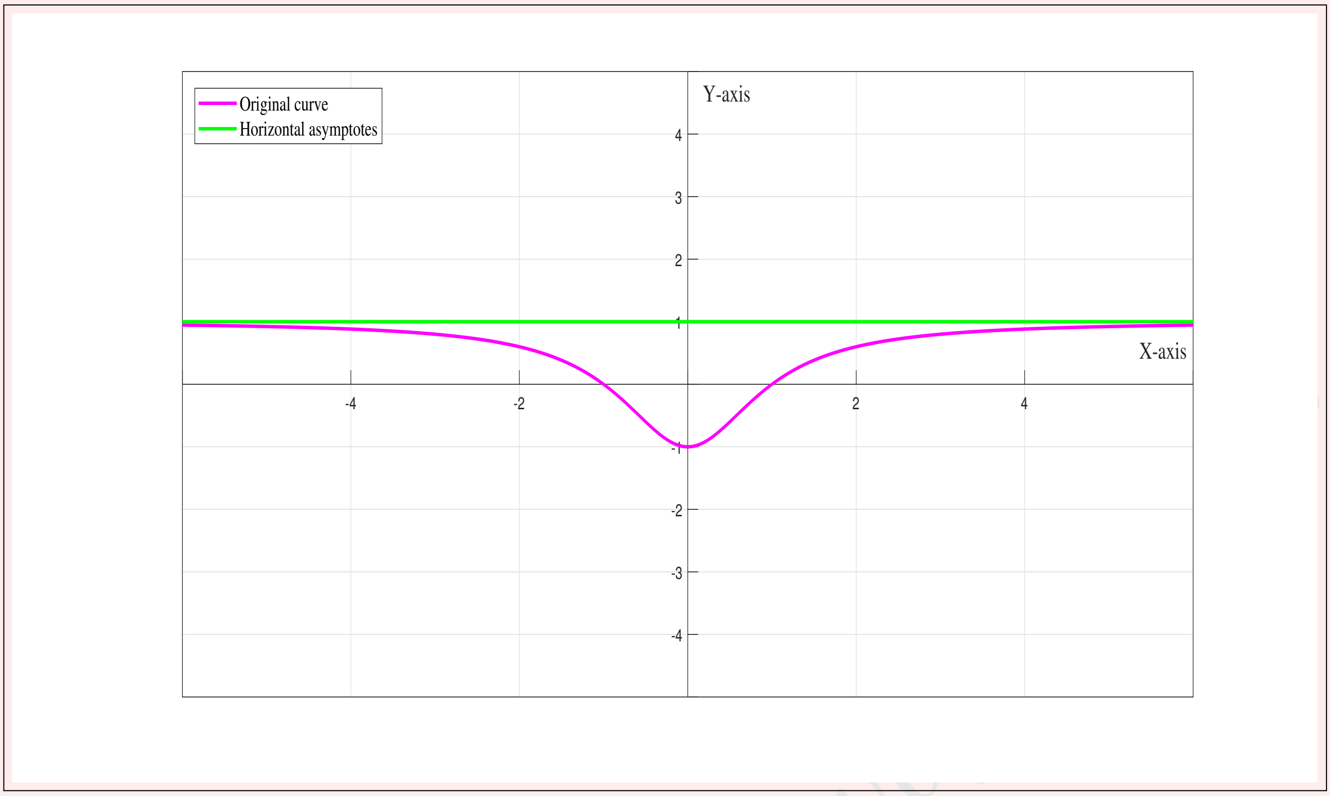


Figure 5: Horizontal Asymptotes.

1. Put $x = 1, y = m$ in $\phi_n(x, y), \phi_{n-1}(x, y), .., \phi_1(x, y)$.
2. Find all the real roots of $\phi_n(m) = 0$.
3. If m is a non-repeated root, then the corresponding value of c is given by $c\phi'_n(m) + \phi_{n-1}(m) = 0, (\phi'_n(m) \neq 0)$.
4. If m is a repeated root occurring twice, then the two values of c are given by $\frac{c^2}{2!}\phi''_n(m) + c\phi'_{n-1}(m) + \phi_{n-2}(m) = 0, (\phi''_n(m) \neq 0)$.

Example 3. *Find all the asymptotes of the curve* $x^3 - x^2y - xy^2 + y^3 + 2x^2 - 4y^2 + 2xy + x + y + 1 = 0$
Solution: *In the curve the highest degree term of x is x^3 and its coefficient is 1. Hence there exists no asymptote parallel to x – axis. Also the coefficient of highest degree term in y is 1, thus the curve has no asymptote parallel to y-axis. Now finding oblique asymptote of the form $y = mx + c$ consider*

$$\phi_3(x, y) = x^3 - x^2y - xy^2 + y^3$$
$$\phi_2(x, y) = 2x^2 - 4y^2 + 2xy$$
$$\phi_1(x, y) = x + y$$

Put $x = 1, y = m$ in above expression and get,

$$\phi_3(m) = m^3 - m^2 - m + 1$$
$$\phi_2(m) = 2 - 4m^2 + 2m$$
$$\phi_1(m) = 1 + m$$

The values of m are obtained by solving $m^3 - m^2 - m + 1 = 0$. This implies $(m^2 - 1)(m - 1) = 0$. Hence $m = 1, 1, -1$.

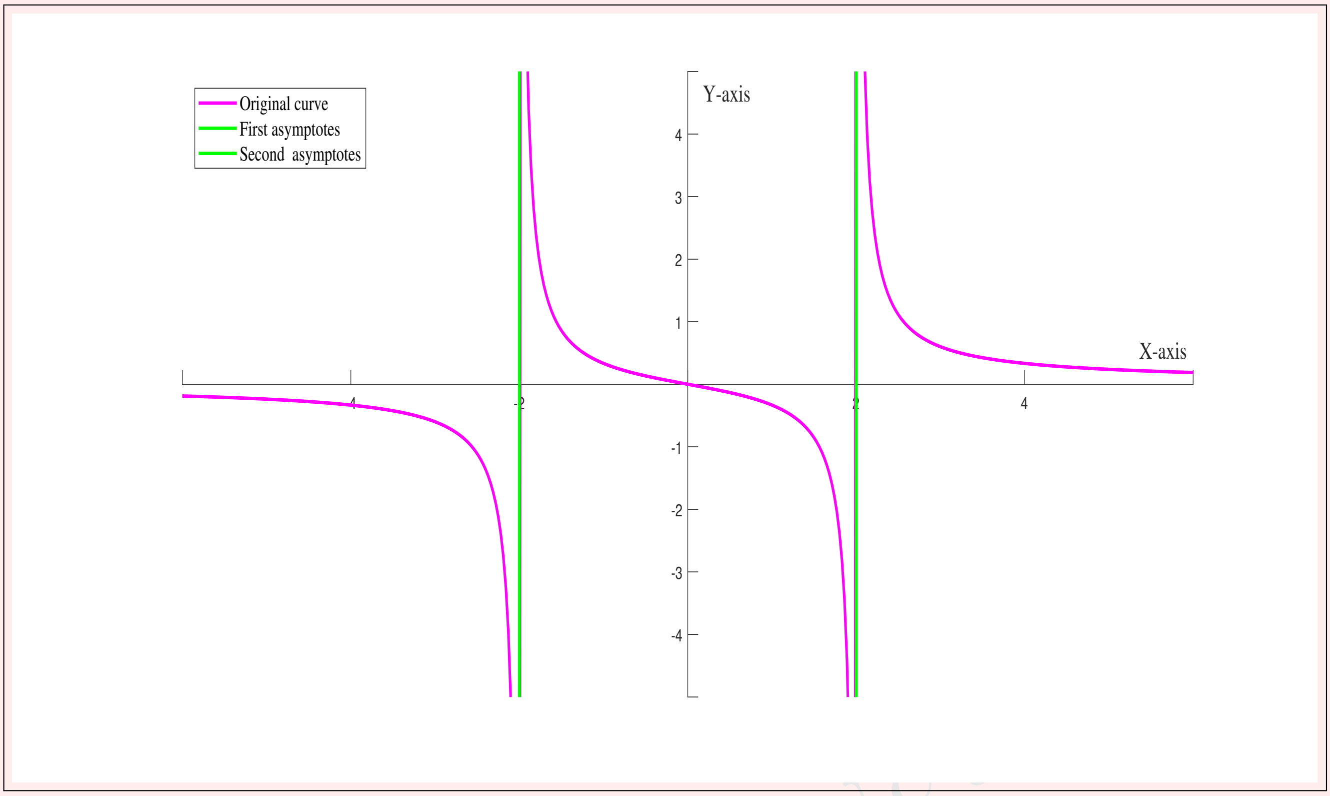


Figure 6: Vertical Asymptotes.

For $m = -1$ *the corresponding value of* c *is given by* $c\phi_3'(m) + \phi_2(m) = 0$.
This gives $c = -\dfrac{\phi_2(m)}{\phi_3'(m)} = 1$.
Hence the asymptote is $y = -x + 1$.
Now for $m = 1$ *(repeated root), the value of* c *is given by* $\dfrac{c^2}{2!}\phi_3''(m) + c\phi_2'(m) + \phi_1(m) = 0$.
This implies $\dfrac{c^2}{2!}(6m - 2) + c(-8m + 2) + (1 + m) = 0 \implies c^2 - 3c + 1 = 0 \implies c = \dfrac{3 \pm \sqrt{5}}{2}$.
Hence the corresponding asymptotes are $y = x + \dfrac{3 \pm \sqrt{5}}{2}$.

5 Envelopes

Assume that a one parameter family of curves can be represented by $f(x, y, \alpha) = 0$. The envelop of the above family satisfy the following equations

$$f(x, y, \alpha) = 0 \quad and \ \frac{\partial f}{\partial \alpha} = 0$$

Example 4. *Find the envelop of the family of the curves given by the equation* $xcos\alpha + ysin\alpha = lcos\alpha sin\alpha$.
Solution: *The given equation can be written as* $\dfrac{x}{sin\alpha} + \dfrac{y}{cos\alpha} = l ......(1)$
Another equation is $-xcosec\alpha cot\alpha + ysec\alpha tan\alpha = 0....(2)$.
Solving (1) and (2) and get, $\tan\alpha = \sqrt[3]{\dfrac{x}{y}}$. *This implies* $\sin\alpha = \dfrac{\sqrt[3]{x}}{\sqrt{x^{\frac{2}{3}} + y^{\frac{2}{3}}}}$ *and* $\cos\alpha = \dfrac{\sqrt[3]{y}}{\sqrt{x^{\frac{2}{3}} + y^{\frac{2}{3}}}}$
Substitute the above values in (1) and get, $(x^{\frac{2}{3}} + y^{\frac{2}{3}})\sqrt{x^{\frac{2}{3}} + y^{\frac{2}{3}}} = l \implies x^{\frac{2}{3}} + y^{\frac{2}{3}} = l^{\frac{2}{3}}$.

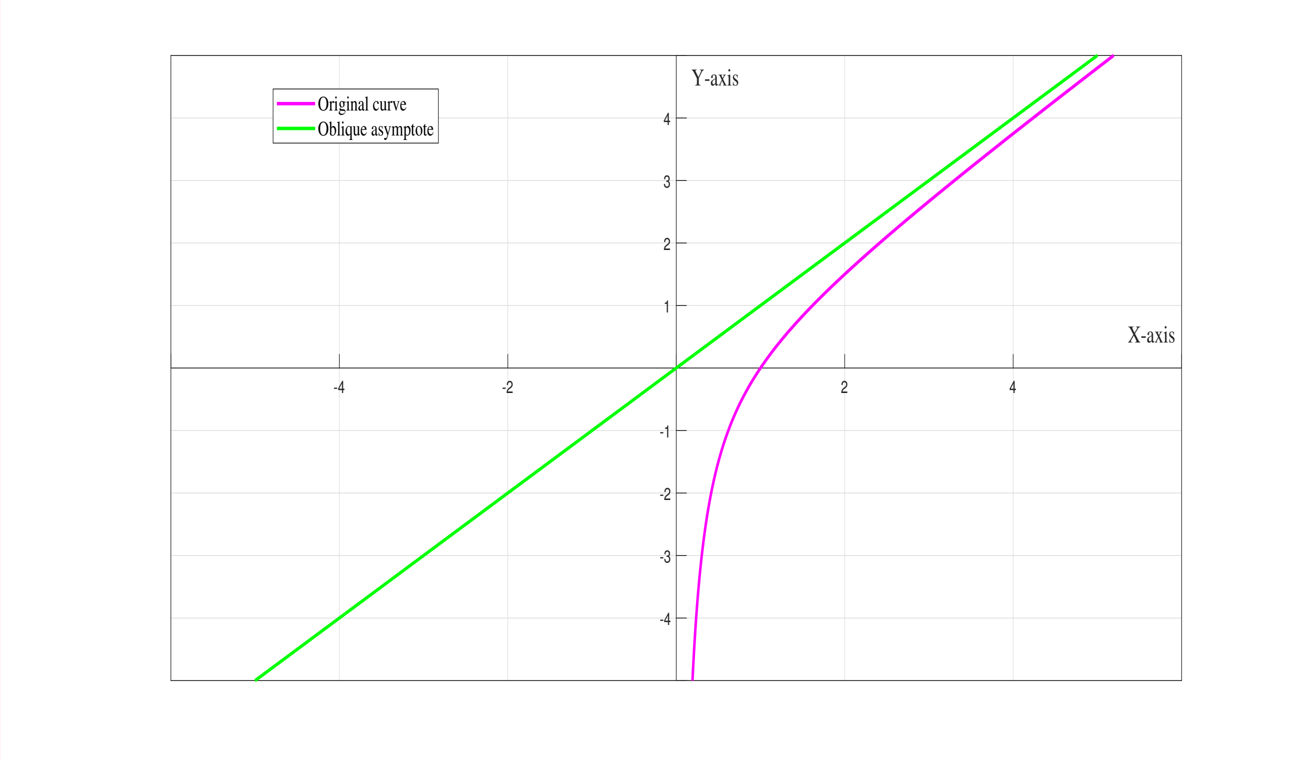


Figure 7: Oblique Asymptotes.

6 Singular points, concavity and inflection points

In the light of what has been said thus far about the meaning of a continuous curve, we may say, without proof, that if a curve is continuous throughout an interval, then, in general, there will exist a tangent to the curve at every point of the curve in that interval. This is true for nearly all curves. There are continuous curves, however, for which no tangent exists. Thus, if RS is an arc of a curve, and if at the point P on RS there exists one and only one tangent, AB, to the curve, then point P is known as an ordinary point of the curve.

Definition 6.1 (Singular points). *A point through which more than one branch of a curve pass is called Multiple point of a curve.*
If n branches passes through the point is called Multiple point of order n. The Multiple points are sometimes called Singular points.

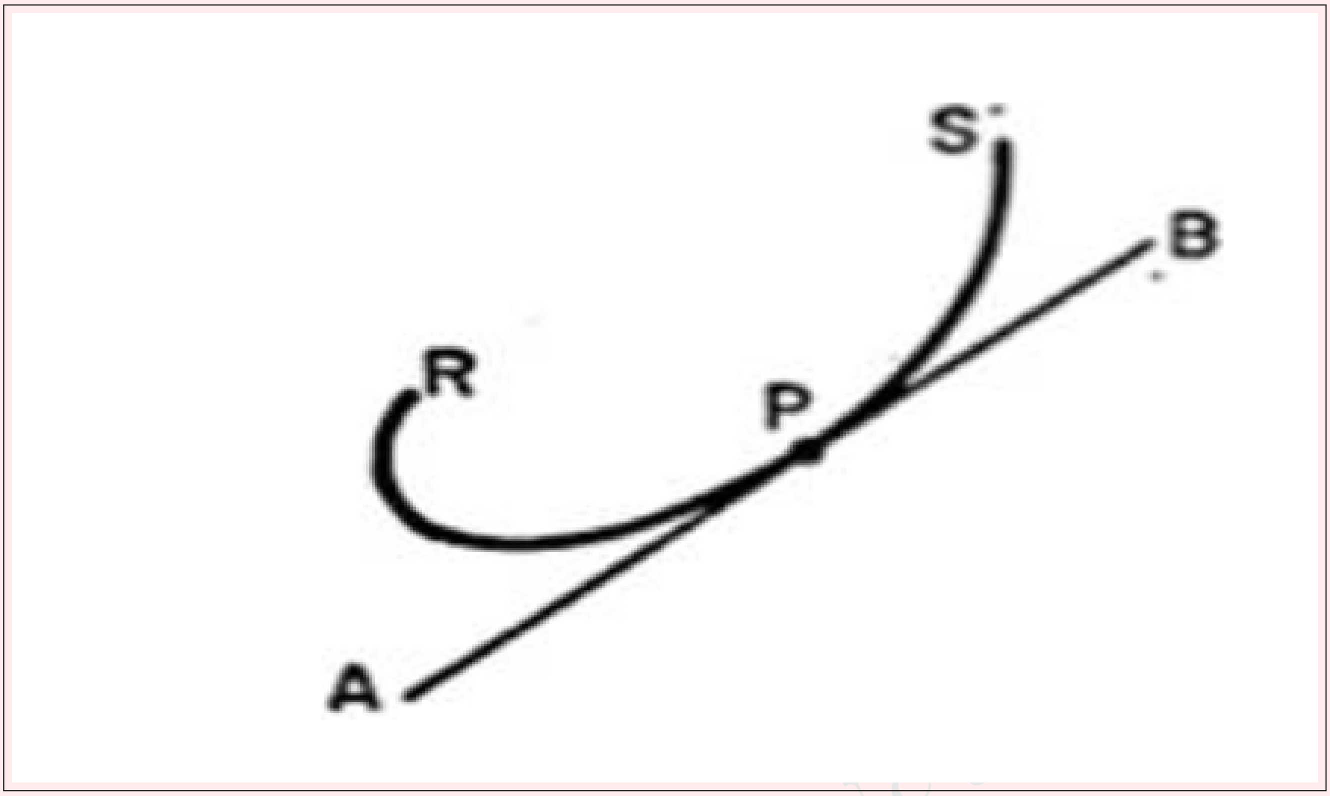
According to the above definition, depending on the number of branches the singular points can be characterize as follows:

(i) **Node**: If at a point P on a curve there exist two distinct tangents or branches, then that point is called a node (See first figure of 8).

(ii) **Cusp**: If the two tangents at a given point are coincide with each other then it is called a cusp (See second and third figures of 8). There are several kinds of cusps:

  (a) if the curve in the neighborhood of a cusp lies partly on one side of the tangent and partly on the other side, the point is known as a cusp of the first kind;

  (b) if the curve lies entirely on one side of the common tangent (in the region of tangency), the point is known as a cusp of the second kind;



(c) if there are two distinct cusps at the same point, it is known as a point of osculation.

(iii) **Isolated or conjugate point**: If there exists no tangents at the corresponding point is called isolated point (See last figure of 8).

All points having two and only two tangents, whether real or imaginary, distinct or coincident, are called double points of the curve. Thus nodes and cusps of all kinds are double points. Triple points are such points on a curve for which there are three tangents; similarly for quadruple points, etc. An isolated point on a curve is also called a conjugate point. All points that are not ordinary points are known as singular points.

6.1 Working rule to determine the type of double point

Let the given curve be $f(x, y) = 0$.
The multiple points are determined by the equations:

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0 \text{ and } f(x, y) = 0$$

To determine the characteristic of the points let us define $D = \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 - \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2}$. Then

1. If D is positive, double point is a node or conjugate point.
2. If $D = 0$, then the double point is a cusp or conjugate point.
3. If D is negative, double point is an isolated point.

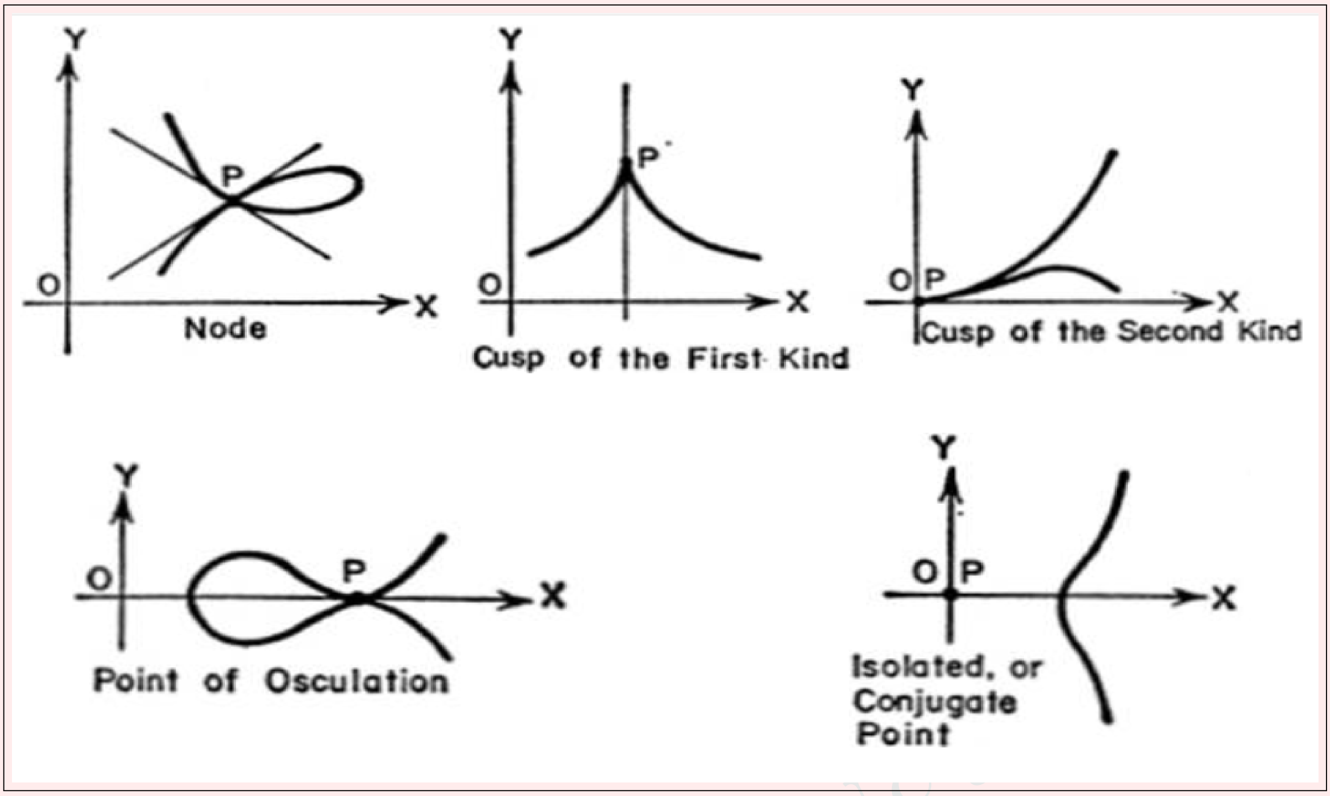


Figure 8: Different type of singular points

7 Curve tracing in Cartesian coordinates, tracing in polar coordinates of standard curves

Recall that by the graph of a function $f : D \to R$ we mean the set of points $\{(x, f(x)) : x \in D\}$. Similarly, the set of points $\{(x, y) : f(x, y) = 0\}$ is known as the graph of the functional relation $f(x, y) = 0$. Graphing a function or a functional relation means showing the points of the corresponding set in a plane. Thus, essentially curve tracing means plotting the points which satisfy a given relation. However, there are some difficulties involved in this. Let's see what these are and how to overcome them.

It is often not possible to plot all the points on a curve. The standard technique is to plot some suitable points and to get a general idea of the shape of the curve by considering tangents, asymptotes, singular points,extreme points, inflection points, concavity, monotonicity, periodicity etc. Then we draw a free hand curve as nearly satisfying the various properties as is possible.

The curves or graphs that we draw have a limitation. If the range of values of either (or both) variable is not finite, then it is not possible to draw the complete graph. In such cases the graph is not only approximate, but is also incomplete. For example, consider the simplest curve, a straight line. Suppose we want to draw the graph $f : R \to R$ such that $f(x) = c$. We know that this is in line parallel to the x-axis. But it is not possible to draw a complete graph as the line has never ending point i.e it has infinite branch.

Let the equation of a curve is $f(x, y) = 0$. We shall now list some steps which, when taken, will simplify our job of tracing this curve.

1. The first step is to determine the extent of the curve. In other words we try to find a region or regions of the plane which cannot any point of the curve. For example, no point on the curve $y^2 = x$, lies in the second or the third quadrant, as the x-coordinate of any point on

the curve has to be non-negative. This means that our curve lies entirely in the first and the fourth quadrants. A point to note here is that it is easier to determine the extent of a curve if its equation can be written explicitly as $y = f(x)$ or $x = f(y)$.

2. The second step is to find out if the curve is symmetrical about any line, or about the origin.

   (a) If all the powers of x occurring in $f(x, y) = 0$ are even, then $f(x, y) = f(-x, y)$ is symmetrical about the $y-$ axis. In this case we need to draw the portion of the graph on only one side of the y-axis. Then we can take its reflection in the y-axis to get the complete graph. We can similarly text the symmetry of a curve about the x-axis.

   (b) If $f(x, y) = 0 \iff f(-x, -y) = 0$, then the curve is symmetrical about the origin is such cases, it is enough to draw the part of the graph above the x-axis and rotate it through 180^0 to get the complete graph.

   (c) If the equation of the curve does not change when we interchange x and y, then the curve is symmetrical about the line $y = x$. The following Table illustrates the application of these criteria for different curves.

Equation	Symmetry
$x^3 + y^2 + y^4 = 0$	About x-axis (Even power of y)
$x^4 + y^2 + y^3 = 0$	About y-axis (Even power of x)
$x^4 + x^2y^2 + y^4 = 0$	About the origin ($f(-x, -y) = 0 \iff f(x, y) = 0$), about both the axes and about the line $y = x$
$x^4 + y^2 = 1$	About both the axes but not about the line $y = x$

Table 1: Some small tricks for algebraic curve.

3. The next step is to determine the points where the curve intersects the axes. If we put $y = 0$ in $f(x, y) = 0$, and solve the resulting equation for x, we get the points of intersection with the x-axis. Similarly, putting $x = 0$ and solving the resulting equation for y, we can find the points of intersection with the y-axis.

4. Try to locate the points where the function is discontinuous.

5. Calculate $\frac{dy}{dx}$. This will help you in locating the portions where the curve is rising $\left(\frac{dy}{dx} > 0\right)$ or falling $\left(\frac{dy}{dx} < 0\right)$ or the points where it has a corner $\left(\frac{dy}{dx} \text{ does not exist}\right)$.

6. Calculate $\frac{d^2y}{dx^2}$. This will help you in locating maxima $\left(\frac{dy}{dx} = 0, \frac{d^2y}{dx^2} < 0\right)$ and minima $\left(\frac{dy}{dx} = 0, \frac{d^2y}{dx^2} > 0\right)$. You will also be able to determine the points of inflection $\left(\frac{d^2y}{dx^2} = 0\right)$. These will give you a good idea about the shape of the curve.

7. The next step is to find the asymptotes, if there are any. They indicate the trend of the branches of the curve extending to infinity.

8. Another important step is to detennine the singular points. The shape of the curve at these points is, generally, more complex, as more than one branch of the curve passes through them.

9. Finally, plot as many points as you can, around the points already plotted. Also try to draw tangents to the curve at some of these plotted points. For this you will have to calculate the derivative as these points. Now join the plotted points by a smooth curve (except at point of discontinuity).

Corollary 7.1.

1. *A curve is symmetrical about a line if, when we fold the curve on the line, the two portions of the curve exactly coincide.*

2. *A curve is symmetrical about the origin if we get the same curve after rotating it through* 180^0.

Example 5. *Let us trace the curve* $(x^2 - 1)(y^2 - 4) = 4$.

Solution: From the given equation it is clear that there exists even power in both the variables x and y. Hence it is symmetric about both the axes. Hence to draw the graph of the given curve we consider only for the first quadrant.

Therefore if we assume that $x > 0, y > 0$ then the it is clear from the given equation that $x > 1, y > 2$.

Equating to zero the coefficients of the highest powers of x and y, we get $y = \pm 2$ and $x = \pm 1$. Hence they are asymptotes of the given curve.

Also $x^2 - 1 = \dfrac{4}{y^2 - 4}$, this implies if x increase then y decrease.

As $x \to \infty, y \to 2$ and similarly, if $y \to \infty, x \to 1$. Then combining all the statements we draw the graph of the given curve as follows:

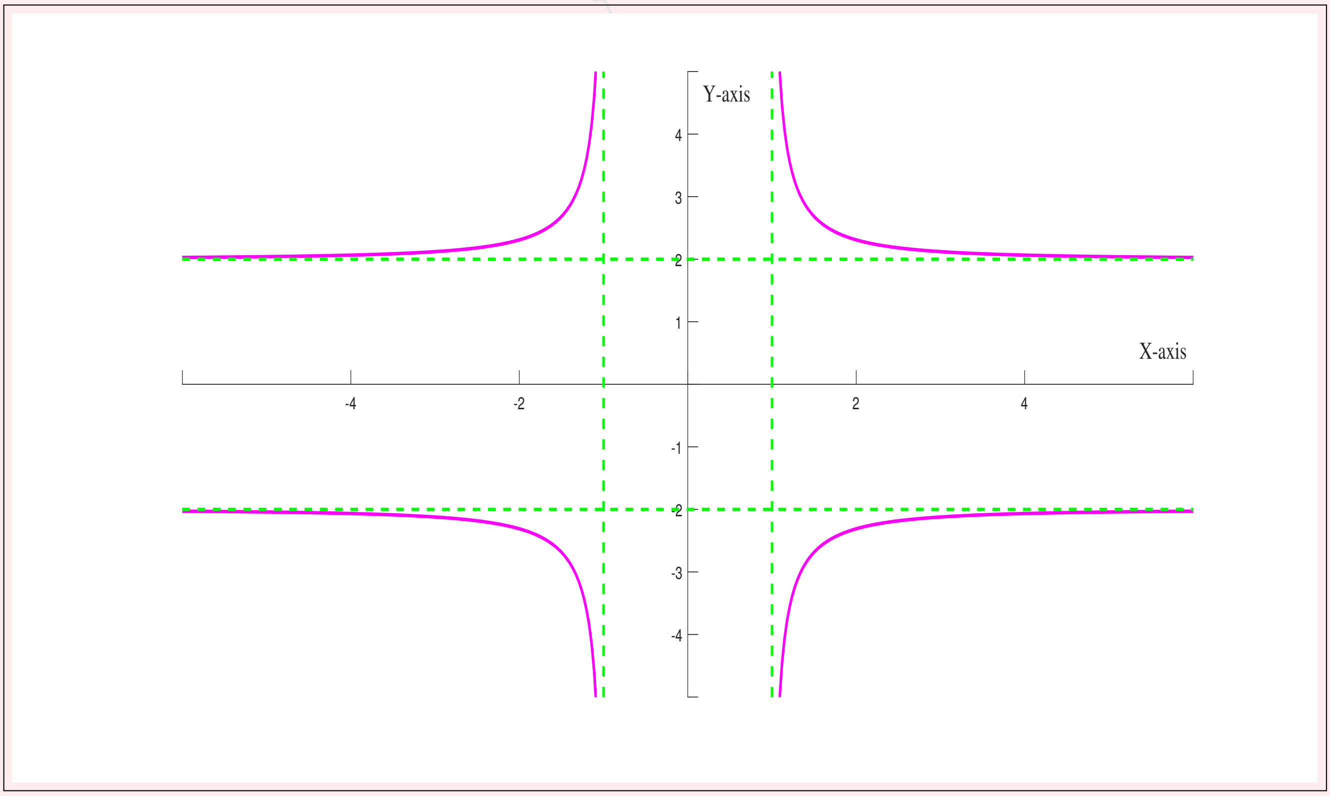


Figure 9: Schematic diagram of the given curve

8 L'Hospital's rule

L'Hospital rule states that when the limit is applied to a fraction of two functions resulting in an indeterminate form then it is equal to the limit of the fraction formed by the individual derivatives of functions. The L'Hospital rule uses derivatives of each function to solve the limit which helps us evaluate the limits which results in an indeterminate form. Now we define what is called indeterminate form as follows:

Definition 8.1. *The indeterminate forms are the forms with two functions whose limits cannot be determined by putting the limits in the function. The indeterminate form is the form that is undefined mathematically. The forms whose value cannot be evaluated by directly applying the limits are called indeterminate forms. Indeterminate form includes* $\frac{0}{0}, \frac{\pm\infty}{\pm\infty}, 0*\infty, \infty-\infty, 1^{\infty}, 0^0$ *etc. The first two are most common forms in which the L'Hospital rule is applied.*

8.1 Formula for L'Hospital's rule

Let $f(x)$ and $g(x)$ be two continuous as well as differentiable functions. If limits x tends to result in an indeterminate form, then the L'Hospital rule is applied and it states

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Proposition 8.2. *For applying the L'Hospital rule the following conditions must be hold.*

- *The functions f, g must be differentiable at a.*
- *The limit of the quotient of the derivatives of a given function should exist.*

Example 6. *Evaluate the following limits:*

1. $\lim_{x\to 0}\frac{sinx - x}{x^2}$
2. $\lim_{x\to 0+}\left(\frac{1}{x^2} - \frac{1}{tanx}\right)$

Solution:

1. As $x \to 0$, both the numerator and denominator approach zero. Therefore, we can apply L'Hôpital's rule.
   We obtain

$$\begin{aligned}\lim_{x\to 0}\frac{sinx - x}{x^2} &= \lim_{x\to 0}\frac{cosx - 1}{2x}\left(\frac{0}{0}\text{ form}\right)\\ &= \lim_{x\to 0}\frac{-sinx}{2}\\ &= 0\end{aligned}$$

   [since both the numerator and denominator goes to 0 as $x \to 0$, therefore applying L'Hôpital's rule]